Chapter 7, Hash tables:

- General introduction; separate chaining (week-before Friday)
- Open addressing (last week Monday)
- Hash functions (last week Wednesday)
- Practice open addressing (last week Thursday lab)
- Perfect hashing (**Today**)
- Hash table wrap-up (Wednesday)
- (Begin Strings in lab Thursday)

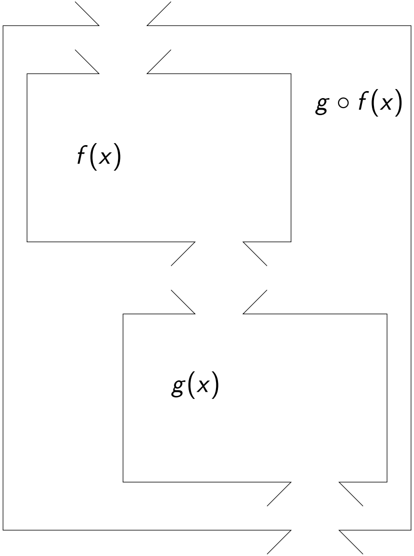
Today:

- Perfect hashing anticipated
  - Motivation
  - Goals
- Perfect hashing accomplished
  - Definition of universal hashing
  - Hash function class $\mathscr{H}_{pm}$
  - Theorems and proofs
- Perfect hashing applied
  - The design of a perfect hashing scheme
  - The given code for the project

A hashing scheme must reduce the occurrence of collisions and “deal” with them when they happen.

- *Separate chaining*, where $m < n$, deals with collisions by chaining keys together in a bucket.
- *Open addressing*, where $n < m$, deals with collisions by finding an alternate location.
- *Perfect hashing* deals with collisions by preventing them altogether.

This topic is parallel with the *optimal BST problem*: What if we knew the keys ahead of time? What if we got to choose the hash function based on what keys we have?



Let $\mathscr{H}$ stand for a *class* of hash functions (a set of hash functions defined by some formula).

Let m be the number of buckets.

$\mathscr{H}$ is *universal* if

$$\forall\, k, \ell \in \mathit{Keys},\ \ |\{h \in \mathscr{H} \mid h(k) = h(\ell)\}| \leq \frac{|\mathscr{H}|}{m}$$

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One particular *family* of *classes* of hash functions, given p, a prime number greater than all keys, and m, the number of buckets, is denoted $\mathscr{H}_{mp}$:

$$\mathscr{H}_{mp} = \{\ h_{ab}(k) = ((ak + b) \mod p) \mod m \mid a \in [1, p) \text{ and } b \in [0, p)\}$$

Theorem $\mathscr{H}_{pm}$ *is universal.*

Proof. *Suppose p and m as specified earlier. Suppose* $k, \ell \in$ *Keys, and* $h_{ab} \in \mathscr{H}_{pm}$ *(which implies supposing that* $a \in [1, p)$ *and* $b \in [0, p)$*).*
Let $r = (a \cdot k + b) \mod p$ *and* $s = (a \cdot \ell + b) \mod p$
Subtracting gives us

$$\begin{aligned} r - s &\equiv (a \cdot k + b) - (a \cdot \ell + b) && \mod p \\ &\equiv a \cdot (k - \ell) && \mod p \end{aligned}$$

Now a cannot be 0 because $a \in [1, p)$*. Similarly* $k - \ell$ *cannot be 0, since* $k \neq \ell$*. Hence* $a \cdot (k - \ell) \neq 0$*.*
Since p is prime and greater than a, k, and ℓ*, it cannot be a factor of* $a \cdot (k - \ell)$*.*
In other words, $a \cdot (k - \ell) \mod p \neq 0$*. By substitution,* $r - s \neq 0$*, and so* $r \neq s$*.*
By another substitution, $(a \cdot k + b) \mod p \neq (a \cdot \ell + b) \mod p$*.*

Define the following function, given k and ℓ, which maps from (a, b) pairs to (r, s) pairs (formally, $[1, p) \times [0, p) \rightarrow [1, p) \times [0, p)$):

$$\phi_{k\ell}(a, b) = ((a \cdot k + b) \mod p, (a \cdot \ell + b) \mod p)$$

Now consider the inverse of that function.

$$\begin{aligned} \phi_{k\ell}^{-1}(r, s) &= (((r - s) \cdot (k - \ell)^{-1}) \mod p), (r - ak) \mod p) \\ &= (a, b) \end{aligned}$$

The existence of ϕ^{-1} implies that ϕ is a one-to-one correspondence. Hence for each (a, b) pair, there is a unique (r, s) pair. Since the pair (a, b) specifies a hash function, that means that for each hash function in the family $\mathscr{H}_{pm}$, there is a unique (r, s) pair.

There are $p-1$ possible choices for a and p choices for b, so there are $p \cdot (p-1)$ hash functions in family $\mathscr{H}_{pm}$. Likewise there are p choices for r, and for each r there are $p-1$ choices for s (since $s \neq r$). Thus we can partition the set $\mathscr{H}_{pm}$ into p subsets by r value, each subset having $p-1$ hash functions.
For a given r, at most one out of every m can have an s that is equivalent to r mod m, in other words, at most $\frac{p-1}{m}$ hash functions.
Now sum that for all p of the subsets of $\mathscr{H}_{pm}$, and we find that the number of hash functions for which k and ℓ collide are

$$p \cdot \frac{p-1}{m} = \frac{p \cdot (p-1)}{m} = \frac{|\mathscr{H}_{pm}|}{m}$$

Therefore $\mathscr{H}_{pm}$ is universal by definition. $\square$

Theorem [Probability of any collisions.] *If Keys is a set of keys, $m = |Keys|^2$, p is a prime greater than all keys, and $h \in \mathscr{H}_{pm}$, then the probability that any two distinct keys collide in h is less than $\frac{1}{2}$.*

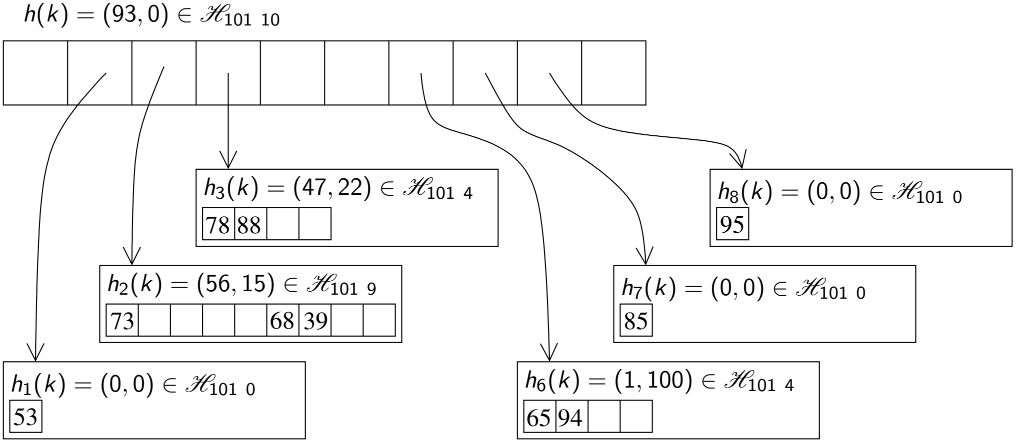
Proof. *Suppose we have a set Keys, $m = |Keys|^2$, p is a prime greater than all keys, and $h \in \mathscr{H}_{pm}$.*
Consider the number of pairs of unique keys. The number of pairs of keys is

$$\binom{n}{2} = \frac{n!}{2! \cdot (n-2)!} = \frac{n!}{2 \cdot (n-2)!} = \frac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{2 \cdot \cancel{(n-2)!}} = \frac{n \cdot (n-1)}{2}$$

Since $\mathscr{H}_{pm}$ is universal, each pair collides with probability $\frac{1}{m}$. Multiply that by the number of pairs, and the expected number of collisions is

$$\begin{aligned}
\frac{n\cdot(n-1)}{2}\cdot\frac{1}{m} &< \frac{n^2}{2}\cdot\frac{1}{m} && \textit{since } n\cdot(n-1) < n^2 \\
&= \frac{n^2}{2}\cdot\frac{1}{n^2} && \textit{since } m = n^2 \\
&= \frac{1}{2} && \textit{by cancelling } n^2
\end{aligned}$$

With the expected number of collisions less than one half, the probability there are any collisions is also less than $\frac{1}{2}$. $\square$


$h(k) = (93, 0) \in \mathscr{H}_{101\ 10}$
$h_3(k) = (47, 22) \in \mathscr{H}_{101\ 4}$
78 88
$h_8(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
95
$h_2(k) = (56, 15) \in \mathscr{H}_{101\ 9}$
73 68 39
$h_7(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
85
$h_1(k) = (0, 0) \in \mathscr{H}_{101\ 0}$
53
$h_6(k) = (1, 100) \in \mathscr{H}_{101\ 4}$
65 94

Coming up:

Do **Open Addressing Hashtable** *project (due Mon, Apr 21)*
Do **Perfect hashing** *project (due Mon, Apr 28)*

Due **Mon, Apr 21** *(end of day)*
Read Sections 7.(4 & 5)
(No practice problems or quiz)

Due **Wed, Apr 23** *(end of day)*
Re-read the last part of Section 7.3
Take quiz

Due **Fri, Apr 25** *(end of day)*
Read Section 8.1
Do Exercises 8.(4 & 5)
Take quiz