Chapter 7 outline:

▶ Introduction, function equality, and anonymous functions (last week Monday)
▶ Image and inverse images (last week Wednesday)
▶ Function properties, composition, and applications to programming (last week Friday)
▶ Cardinality (Monday)
▶ Practice quiz and Countability (Today)
▶ Review (Monday, Apr 18)
▶ Test 3, on Ch 6 & 7 (Wednesday, Apr 20)
Two finite sets \( X \) and \( Y \) have the *the same cardinality* as each other if there exists a one-to-one correspondence from \( X \) to \( Y \).

To use this *analytically*: 

Suppose \( X \) and \( Y \) have the same cardinality. Then let \( f \) be a one-to-one correspondence from \( X \) to \( Y \). 

\( f \) is both onto and one-to-one.

To use this *synthetically*: 

*Given sets \( X \) and \( Y \)...*

[Define \( f \)] Let \( f : X \to Y \) be a function defined as ... 

Suppose \( y \in Y \). *Somehow find \( x \in X \) such that \( f(x) = y \). Hence \( f \) is onto.* 

Suppose \( x_1, x_2 \in X \) such that \( f(x_1) = f(x_2) \). *Somehow show \( x_1 = x_2 \). Hence \( f \) is one-to-one.* 

Since \( f \) is a one-to-one correspondence, \( X \) and \( Y \) have the same cardinality as each other.
A finite set $X$ has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from \{1, 2, \ldots n\} to $X$. Moreover, $|\emptyset| = 0$. 
Two finite sets $X$ and $Y$ have the *the same cardinality* as each other if there exists a one-to-one correspondence from $X$ to $Y$.

A finite set $X$ has cardinality $n \in \mathbb{N}$, which we write as $|X| = n$, if there exists a one-to-one correspondence from $\{1, 2, \ldots n\}$ to $X$. Moreover, $|\emptyset| = 0$.

Given a set $X$, if there exists $n \in \mathbb{N}$ and a one-to-one correspondence from $\{1, 2, \ldots n\}$ to $X$, then $X$ is *finite* and $|X| = n$. Otherwise, $X$ is *infinite*. 
Are all infinities equal?

Which is more intuitive to you,

\[ |N| = |W| = |Z| = |Q| = |R| \]

or

\[ |N| < |W| < |Z| < |Q| < |R| \]
Thm 7.19. $\mathbb{W}$ and $\mathbb{N}$ have the same cardinality.

Proof. [We need a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{W}$.]
Let $f : \mathbb{W} \to \mathbb{N}$ be defined so that $f(n) = n + 1$.
Suppose $n \in \mathbb{N}$. Then $f(n - 1) = (n - 1) + 1 = n$, so $f$ is onto.
Next suppose $n_1, n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then $n_1 + 1 = n_2 + 1$, and moreover $n_1 = n_2$. Hence $f$ is one-to-one.
Since a one-to-one correspondence exists between $\mathbb{W}$ and $\mathbb{N}$, the two sets have the same cardinality. \(\Box\)

A set $X$ is *countably infinite* if there exists a one-to-one correspondence from $\mathbb{N}$ to $X$.
A set is *countable* if it is finite or countably infinite. Otherwise it is *uncountable*. 
Thm 7.20. \( \mathbb{Z} \) is countably infinite.

Proof. [We need a one-to-one correspondence from \( \mathbb{N} \) to \( \mathbb{Z} \).]

Let \( f : \mathbb{N} \to \mathbb{Z} \) be defined so that

\[
 f(x) = \begin{cases} 
 n \div 2 & \text{if } n \text{ is even} \\
 -(n \div 2) & \text{otherwise}
\end{cases}
\]

Since \( f \) is a one-to-one correspondence, \( \mathbb{Z} \) is countably infinite. \( \Box \)
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fun findRoom(busNum, seatNum) =
  let
    fun nextPair(a, b) =
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)
      else if b = 1 andalso a mod 2 = 0
        then (a + 1, 1)
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)
      else (a - 1, b + 1);
    fun findRoomHelper(i, currentPair) =
      if currentPair <> (busNum, seatNum)
        then findRoomHelper(i + 1, nextPair(currentPair))
      else i;
  in
    findRoomHelper(1, (1, 1))
  end;
fun findBusSeat(room) =
  let
    fun nextPair(a, b) =
      if a = 1 andalso b mod 2 = 1 then (1, b + 1)
      else if b = 1 andalso a mod 2 = 0
      then (a + 1, 1)
      else if (a + b) mod 2 = 1 then (a + 1, b - 1)
      else (a - 1, b + 1);
    fun findBusSeatHelper(i, currentPair) =
      if i <> room
      then findBusSeatHelper(i + 1,
        nextPair(currentPair))
      else currentPair;
  in
    findBusSeatHelper(1, (1, 1))
  end;
**Thm 7.21.** $\mathbb{Q}^+$ is countably infinite.

So,

$$|\mathbb{N}| = |\mathbb{W}| = |\mathbb{Z}| = |\mathbb{Q}|$$

What about $\mathbb{R}$?
Thm 7.22. (0, 1) has the same cardinality as $\mathbb{R}$. 

\begin{center}
\begin{tabular}{c|c|c|c|c|c}
 0 & .25 & .5 & .75 & 1 \\
\end{tabular}
\end{center}
Thm 7.23. $(0, 1)$ is uncountable.

Proof. Suppose $(0, 1)$ is countable. Then there exists a one-to-one correspondence $f : \mathbb{N} \to (0, 1)$. We will use $f$ to give names to the all the digits of all the numbers in $(0, 1)$, considering each number in its decimal expansion, where each $a_{i,j}$ stands for a digit.: 

\begin{align*}
f(1) &= 0.a_{1,1}a_{1,2}a_{1,3} \ldots a_{1,j} \ldots \\
f(2) &= 0.a_{2,1}a_{2,2}a_{2,3} \ldots a_{2,j} \\
\vdots \\
f(x) &= 0.a_{x,1}a_{x,2}a_{x,3} \ldots a_{x,j} \\
\vdots 
\end{align*}

Now construct a number $d = 0.d_1d_2d_3 \ldots d_i \ldots$ as follows

\[ d_i = \begin{cases} 
1 & \text{if } a_{i,i} \neq 1 \\
2 & \text{if } a_{i,i} = 1 
\end{cases} \]
Since $d \in (0, 1)$ and $f$ is onto, there exists an $x \in \mathbb{N}$ such that $f(x) = d$. Moreover,

$$f(x) = 0.a_{x,1}a_{x,2}a_{x,3}\ldots a_{x,x}\ldots$$

so

$$d = 0.a_{x,1}a_{x,2}a_{x,3}\ldots a_{x,x}\ldots$$

by substitution. In other words, $d_i = a_{x,i}$, and specifically $d_x = a_{x,x}$. However, by the way that we have defined $d$, we know that $d_x \neq a_{x,x}$, a contradiction. Therefore $(0, 1)$ is not countable. □