Chapter 7 in context:

- Chapter 5 Relations: Builds on proofs about sets
- Chapter 6 Self Reference: Interlude between Chapters 5 and 7, focuses on recursive thinking
- Chapter 7 Function: Builds on proofs about relations

Chapter 7 outline:

- Introduction, function equality, and anonymous functions (Today)
- Image and inverse images (next week Monday)
- Function properties, composition, and applications to programming (next week Wednesday)
- Cardinality (next week Friday)
- Countability (Monday, Nov 22)
- Review (Monday, Nov 29)
- Test 3, on Ch 6 & 7 (Wednesday, Dec 1)
Cross out the term/concept that was not used in the reading for today as a way to think about functions

A kind of machine  
A mapping between two collections  
A kind of relation  
A form of induction

For the function \( f : X \rightarrow Y \), \( X \) is the _______________ and \( Y \) is the _______________.

_____________  
function  
codomain  
constant  
first-class value  
domain  
relation
function

input, raw materials, parameters

output, result, returned value
<table>
<thead>
<tr>
<th>Name</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>x3498</td>
</tr>
<tr>
<td>Bob</td>
<td>x4472</td>
</tr>
<tr>
<td>Carol</td>
<td>x5392</td>
</tr>
<tr>
<td>Dave</td>
<td>x9955</td>
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<tr>
<td>Eve</td>
<td>x2533</td>
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<td>Fred</td>
<td>x9448</td>
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<tr>
<td>Georgia</td>
<td>x3684</td>
</tr>
<tr>
<td>Herb</td>
<td>x8401</td>
</tr>
</tbody>
</table>
\[ y = 4 - x^2 \]

\[ y^2 = 4 - x^2 \]
Not a function. (There's a domain element that is related to two things.)

Not a function. (There's a domain element that is not related to anything.)

A function. (It's OK that two domain elements are related to the same thing and one codomain element has nothing related to it.)
Definition of function

Informal: A function is a relation in which everything in the first set is related to exactly one thing in the second set.

Formal: $f \subseteq X \times Y$ is a function if

$$
\forall x \in X, \quad \exists y \in Y \mid (x, y) \in f
$$

existence of $y$

$$
\land \forall y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2
$$

uniqueness of $y$
Change of notation

Informal: A \textit{function} is a relation in which everything in the first set is related to \textit{exactly one thing} in the second set.

Formal (relation notation): \( f \subseteq X \times Y \) is a \textit{function} if

\[
\forall x \in X, \quad \exists y \in Y \mid (x, y) \in f
\]

\[
\land \forall y_1, y_2 \in Y, ((x, y_1), (x, y_2) \in f) \rightarrow y_1 = y_2
\]

existence of \( y \)

uniqueness of \( y \)

Formal (function notation): \( f \subseteq X \times Y \) is a \textit{function} if

\[
\forall x \in X, \quad \exists y \in Y \mid f(x) = y
\]

\[
\land \forall y_1, y_2 \in Y, (f(x) = y_1 \land f(x) = y_2) \rightarrow y_1 = y_2
\]

existence of \( y \)

uniqueness of \( y \)

We call \( X \) the \textit{domain} and \( Y \) the \textit{codomain} of \( f \).
Definition of function equality. Let \( f, g : X \rightarrow Y \)

Old definition: functions are sets.

\[
f = g \text{ if } \forall f \subseteq g \land g \subseteq f
\]

New definition: based on function notation.

\[
f = g \text{ if } \forall x \in X, f(x) = g(x)
\]
Function equality: \( f = g \) if \( \forall \ x \in X, f(x) = g(x) \)

Let \( f, g : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = x \cdot (x - 1) - 6 \) and \( g(x) = (x - 3)(x + 2) \).

Prove \( f = g \).
The old and new definitions of function equality are equivalent.

**Ex 7.2.1.** \((\forall x \in X, f(x) = g(x))\) iff \((f \subseteq g \land g \subseteq f)\).
The old and new definitions of function equality are equivalent.

**Ex 7.2.1.** \((\forall x \in X, f(x) = g(x))\) iff \((f \subseteq g \land g \subseteq f)\).

**Proof.** First, suppose \(\forall x \in X, f(x) = g(x)\), that is, \(f = g\) by definition of function equality. Further suppose \((x, y) \in f\). By function notation, \(f(x) = y\). By supposition and substitution, \(g(x) = y\). By relation notation, \((x, y) \in g\). Finally, \(f \subseteq g\) by definition of subset.

Similarly \(f \subseteq g\), and therefore \(f = g\) by definition of set equality.

Conversely, suppose \(f \subseteq g \land g \subseteq f\), that is, \(f = g\) by definition of set equality. Further suppose \(x \in X\).

Let \(y = f(x)\). Note that this \(y \in Y\) must exist by definition of function. By relation notation, \((x, y) \in f\).

By definition of subset [or set equality], \((x, y) \in g\). In function notation, that is \(g(x) = y\), and so \(f(x) = g(x)\) by substitution. Therefore \(f = g\) by definition of function equality. \(\square\)
For next time:

Pg 331: 7.2.(2 & 3)
Pg 335: 7.3.(3, 4, 8)

Read 7.4
Skim 7.5