Chapter 7 outline:

- Introduction, function equality, and anonymous functions (last week Friday)
- Image and inverse images (Monday)
- Function properties, composition, and applications to programming (Today)
- Cardinality (Friday)
- Countability (next week Monday)
- Review (Monday after Thanksgiving, Nov 27)
- Test 3, on Ch 6 & 7 (Wednesday after Thanksgiving, Nov 29)

Today:

- Programming: map and filter ✓ — did it last time
- Definition of one-to-one and onto, plus proofs
- Inverse functions
- Definition of function composition, plus proofs
Not a function.
(There’s a domain element that is related to two things.)

Not a function.
(There’s a domain element that is not related to anything.)
Onto (Surjection)

Everything in the codomain is hit.

\[ f : X \rightarrow Y \] is onto if \( \forall y \in Y, \exists x \in X \mid f(x) = y. \]

**Analytic use:**
\( f \) is onto.
\( y \in Y. \)
Hence \( \exists x \in X \) such that \( f(x) = y. \)

**Synthetic use:**
Suppose \( y \in Y. \)
\( \therefore (Somehow find x such that f(x) = y.) \)
Therefore \( f \) is onto.
One-to-one (Injection)

Domain elements don't share.

\[ f \text{ is one-to-one if } \forall x_1, x_2 \in X, \]
\[ \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2. \]

**Analytic use:**
\[ f \text{ is one-to-one.} \]
\[ f(x_1) = f(x_2). \]
Hence \( x_1 = x_2. \)

**Synthetic use:**
Suppose \( x_1, x_2 \in X \) and \( f(x_1) = f(x_2). \)
\[ \therefore \]
\[ (Somehow \ show \ x_1 = x_2.) \]
Therefore \( f \) is one-to-one.
Onto (not one-to-one)  
$|X| \geq |Y|$

One-to-one (not onto)  
$|X| \leq |Y|$

Both onto and one-to-one  
$|X| = |Y|$
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is $f$ one-to-one? Is it onto?

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**Proof.** Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. Then, by how $f$ is defined, $x_1^2 = x_2^2$ which implies $x_1 = x_2$. Therefore, $f$ is one-to-one by definition. □

**Proof.** Suppose $y \in \mathbb{R}$. [Want $x$ such that $f(x) = y$.] Let $x = 2y$. Then $f(x) = 2y^2 = y$. Therefore, $f$ is onto by definition. □
Let \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = \frac{x}{2} \). Is \( f \) one-to-one? Is it onto?

\( f \) is one-to-one.

**Proof.** Suppose \( x_1, x_2 \in \mathbb{R} \) such that \( f(x_1) = f(x_2) \). [Want \( x_1 = x_2 \)] Then, by how \( f \) is defined,
Let \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = \frac{x}{2} \). Is \( f \) one-to-one? Is it onto?

\( f \) is one-to-one.

**Proof.** Suppose \( x_1, x_2 \in \mathbb{R} \) such that \( f(x_1) = f(x_2) \). \([\text{Want } x_1 = x_2]\) Then, by how \( f \) is defined,

\[
\frac{x_1}{2} = \frac{x_2}{2} \\
\therefore x_1 = x_2
\]
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is $f$ one-to-one? Is it onto?

$f$ is one-to-one.

Proof. Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how $f$ is defined,

\[
\frac{x_1}{2} = \frac{x_2}{2} \\
x_1 = x_2
\]

Therefore $f$ is one-to-one by definition. □

$f$ is onto.
Let \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = \frac{x}{2} \). Is \( f \) one-to-one? Is it onto?

\( f \) is one-to-one.

**Proof.** Suppose \( x_1, x_2 \in \mathbb{R} \) such that \( f(x_1) = f(x_2) \). [Want \( x_1 = x_2 \)] Then, by how \( f \) is defined,

\[
\begin{align*}
\frac{x_1}{2} &= \frac{x_2}{2} \\
x_1 &= x_2
\end{align*}
\]

Therefore \( f \) is one-to-one by definition. \( \square \)

\( f \) is onto.

**Proof.** Suppose \( y \in \mathbb{R} \). [Want \( x \) such that \( f(x) = y \)].
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \frac{x}{2}$. Is $f$ one-to-one? Is it onto?

$f$ is one-to-one.

**Proof.** Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. [Want $x_1 = x_2$] Then, by how $f$ is defined,

\[
\frac{x_1}{2} = \frac{x_2}{2} \\
x_1 = x_2
\]

Therefore $f$ is one-to-one by definition. □

$f$ is onto.

**Proof.** Suppose $y \in \mathbb{R}$. [Want $x$ such that $f(x) = y$.]

Let $x = 2y$. Then

\[
f(x) = \frac{2y}{2} = y
\]

Therefore $f$ is onto by definition □
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Is $f$ one-to-one? Is it onto?

$f$ is not one-to-one. $f(2) = 2^2 = 4$ and $f(-2) = (-2)^2 = 4$. Therefore, there exists more than one element in the domain that maps to the same element in the codomain.

$f$ is not onto. Let $y = -1$. There does not exist an $x \in \mathbb{R}$ such that $f(x) = -1$. The range of $f$ is $[0, \infty)$.
Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Is $f$ one-to-one? Is it onto?

$f$ is not one-to-one.
$f(2) = 2^2 = 4$
$f(-2) = (-2)^2 = 4$

$f$ is no onto.
Let $y = -1$.
$\forall x \in \mathbb{R}$ such that $f(x) = -1$. 
**Ex 7.6.4.** If $A \subseteq X$ and $f$ is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

(Ex 7.4.9 was, Prove $A \subseteq F^{-1}(F(A))$, and Ex 7.4.10 was, Find a counterexample for $A = F^{-1}(F(A))$.)
Ex 7.6.4. If $A \subseteq X$ and $f$ is one-to-one, then $F^{-1}(F(A)) \subseteq A$.

(Ex 7.4.9 was, Prove $A \subseteq F^{-1}(F(A))$, and Ex 7.4.10 was, Find a counterexample for $A = F^{-1}(F(A))$.)
Ex 7.6.5. If $A \subseteq Y$ and $f$ is onto, then $A \subseteq F(F^{-1}(A))$. 
Inverse relation: \( R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\} \)

Since a function is a relation, a function has an inverse, but we don’t know that the inverse of a function is a function.

If \( f : X \to Y \) is a one-to-one correspondence, then

\[
f^{-1} : Y \to X = \{(y, x) \in Y \times X \mid f(x) = y\}
\]

is the inverse function of \( f \).

**Theorem 7.8** If \( f : X \to Y \) is a one-to-one correspondence, then \( f^{-1} : Y \to X \) is well defined.

**Proof.** Suppose \( y \in Y \). Since \( f \) is onto, there exists \( x \in X \) such that \( f(x) = y \). Hence \( (y, x) \in f^{-1} \) or \( f^{-1}(y) = x \).

Further suppose \( (y, x_1), (y, x_2) \in f^{-1} \) (That is, suppose that both \( f^{-1}(y) = x_1 \) and \( f^{-1}(y) = x_2 \).) Then \( f(x_1) = y \) and \( f(x_2) = y \). Since \( f \) is one-to-one, \( x_1 = x_2 \).

Therefore, by definition of function, \( f^{-1} \) is well defined. \( \square \)
Relation composition: If $R$ is a relation from $X$ to $Y$ and $S$ is a relation from $Y$ to $Z$, then $S \circ R$ is the relation from $X$ to $Z$ defined as

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

$$g \circ f = \{(x, z) \in X \times Z \mid z = g(f(x))\}$$

**Theorem 7.9** If $f : X \to Y$ and $g : Y \to Z$ are functions, then $g \circ f : X \to Z$ is well defined.

**Proof.** Suppose $x \in X$. Since $f$ is a function, there exists a $y \in Y$ such that $f(x) = y$. Since $g$ is a function, there exists a $z \in Z$ such that $g(y) = z$. By definition of composition, $(x, z) \in g \circ f$, or $g \circ f(x) = z$.

Next suppose $(x, z_1), (x, z_2) \in g \circ f$, or $g \circ f(x) = z_1$ and $g \circ f(x) = z_2$. By definition of composition, there exist $y_1, y_2$ such that $f(x) = y_1$, $f(x) = y_2$, $g(y_1) = z_1$, and $g(y_2) = z_2$. Since $f$ is a function, $y_1 = y_2$. Since $g$ is a function, $z_1 = z_2$.

Therefore, by definition of function, $g \circ f$ is well defined. □
Function composition: If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$ is defined as

$$g \circ f = \{(x, z) \in X \times Z \mid x = g(f(x))\}$$

Let $f(x) = 3x$
Let $g(x) = x + 7$

Then

$$g \circ f(x) = f(x) + 7 = 3x + 7$$
Ex 7.8.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

**Proof.** Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.
Ex 7.8.4. If \( f : X \to Y \) and \( g : Y \to Z \) are both onto, then \( g \circ f \) is onto.

**Proof.** Suppose \( f : X \to Y \) and \( g : Y \to Z \) are both onto.

*Now, we want to prove “ontoness.” Of which function?*
Ex 7.8.4. If \( f : X \to Y \) and \( g : Y \to Z \) are both onto, then \( g \circ f \) is onto.

Proof. Suppose \( f : X \to Y \) and \( g : Y \to Z \) are both onto.

[Now, we want to prove “ontoness.” Of which function? \( g \circ f \). How do we prove ontoness?]
Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

**Proof.** Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$?]

\[\begin{tikzpicture}
\path [draw] (0,0) circle [radius=2cm] (3,0) circle [radius=2cm] (-3,0) circle [radius=2cm];
\draw[->,thick] (-3,0) to node [above] {$X$} (0,0);
\draw[->,thick] (0,0) to node [above] {$Y$} (3,0);
\draw[->,thick] (3,0) to node [above] {$Z$} (-3,0);
\end{tikzpicture}\]
Ex 7.8.4. If $f : X \to Y$ and $g : Y \to Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$? $Z$.]

Further suppose $z \in Z$. [We need to come up with something in the domain of $g \circ f$ that hits $z$. The domain is $X$. We will use the fact that $f$ and $g$ are both onto.]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \xrightarrow{g} & Z \\
\end{array}
\]
Ex 7.8.4. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto, then $g \circ f$ is onto.

Proof. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both onto.

[Now, we want to prove “ontoness.” Of which function? $g \circ f$. How do we prove ontoness? We pick something from the codomain of the function we’re proving to be onto and show that it is hit. What is the codomain of $g \circ f$? $Z$.]

Further suppose $z \in Z$. [We need to come up with something in the domain of $g \circ f$ that hits $z$. The domain is $X$. We will use the fact that $f$ and $g$ are both onto.]

By definition of onto, there exists $y \in Y$ such that $g(y) = z$. Similarly there exists $x \in X$ such that $f(x) = y$. Now,

$$g \circ f(x) = g(f(x)) \quad \text{by definition of function composition}$$

$$= g(y) \quad \text{by substitution}$$

$$= z \quad \text{by substitution}$$

Therefore $g \circ f$ is onto by definition. $\square$
Ex 7.8.5. If $f : X \rightarrow Y$, $g : X \rightarrow Y$ and $h : Y \rightarrow Z$, $h$ is one-to-one, and $h \circ f = h \circ g$, then $f = g$. 
For next time:

Pg 346: 7.6.(2, 3, 6)
Ex “7.5.(a-c)” on Schoology
Pg 351: 7.8.(1, 5, 6)

Skim 7.9
Take last quiz