Chapter 5 roadmap:

- Introduction to relations (previous week Monday)
- Properties of relations (previous week Wednesday and Friday)
- Transitive closure (last week Friday)
- Partial order relations (Today)
- Review for Test 2 (Wednesday)
- Test 2 on Chapters 4 & 5 (Friday)

Today:

- Antisymmetry
- Partial order relations
- Topological sort
Chinnereth
Arnon
Jabbok
Jordan
Dead
Mediterranean

Jabbok → Arnon → Jordan → Dead → Chinnereth → Mediterranean

0 1 2
3 4 5
6 7 8
9 10 11
... ... ...
**Symmetric**
All arrows have a back arrow.

**Asymmetric**
(not symmetric)
There exists an arrow without a back arrow.

**Antisymmetric**
(“very” not symmetric)
No arrows have back arrows except self loops.
Formal definition:
A relation \( R \) on a set \( X \) is antisymmetric if \( \forall x, y \in X, \) if \( (x, y) \in R \) and \( (y, x) \in R \), then \( x = y \).

Informal definition:
If both an arrow and its reverse exist in an antisymmetric relation \( R \), then that arrow must be a self loop (and, hence, it is its own reverse).

Alternate formal definition:
A relation \( R \) on a set \( X \) is antisymmetric if \( \forall (x, y) \in R \), either \( x = y \) or \( (y, x) \notin R \).
A relation $R$ on a set $X$ is antisymmetric if $\forall x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Ex 5.8.9. Prove that $|$ (divides) on $\mathbb{N}$ is antisymmetric.

Proof. Suppose $x, y \in \mathbb{N}$, $x | y$, and $y | x$ (that is, $(x, y), (y, x) \in |$). By definition of divides, there exists $i, j \in \mathbb{N}$ such that

$$x = i \cdot y$$
$$y = j \cdot x$$

Then

$$x = i \cdot j \cdot x \quad \text{by substitution}$$
$$1 = i \cdot j \quad \text{by cancellation}$$
$$i = j = 1 \quad \text{by arithmetic}$$
$$x = y \quad \text{by identity}$$

Therefore $|$ is antisymmetric by definition. $\square$
Antisymmetry:
A relation \( R \) on a set \( X \) is **antisymmetric** if \( \forall x, y \in X, (x, y) \in R \) and \((y, x) \in R\), then \( x = y \).

Partial order relation:
A **partial order relation** (or just **partial order**) is a relation that is reflexive, transitive, and antisymmetric.
A **strict partial order (relation)** is a relation that is irreflexive, transitive and antisymmetric.

Partially ordered set:
A **partially ordered set** or **poset** is a set together with a partial order on that set.
\[ R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\} \]
Comparable: \( a \leq c, d \leq f, e \leq f, e \leq h, c \leq i \)

Not comparable: \( a \) and \( b \); \( d \) and \( e \); \( f \) and \( h \)

Maximal and greatest: \( i \)

Minimal: \( a \) and \( b \)

No least

Everyday examples: Preparing a meal, writing a term paper, getting dressed
A partial order $R$ on a set $X$ is a \textit{total order} if for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$, that is, $x$ and $y$ are comparable.

Standard example of a total order: $\preceq$. 
A *partial order relation* (or just *partial order*) is a relation that is reflexive, transitive, and antisymmetric.

A partial order $R$ on a set $X$ is a *total order* if for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$, that is, $x$ and $y$ are comparable.

A *topological sort* of a partial order $R$ is a total order that is a superset of $R$.

$\mid$ (divides) $\leq$

is prerequisite for Ralph takes before can put on before you put on before
\[ R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\} \]

A topological sort for \( R \): \( R \cup \{(b, c)\} \), written as \( a, b, c, d \)

Another topological sort for \( R \): \( R \cup \{(c, b)\} \), written as \( a, c, b, d \)
For next time:

Pg 226: 5.8.(1-5)
Pg 231 5.9.(1 & 8)
Read 6.(1–3)