Chapter 5 roadmap:
▶ Introduction to relations (Monday before break)
▶ Properties of relations (Wednesday and Friday before break)
▶ Transitive closure (Monday)
▶ Partial order relations (Today)
▶ Review for Test 2 (Friday)
▶ Test 2 on Chapters 4 & 5 (next week Monday)

Today:
▶ Antisymmetry
▶ Partial order relations
▶ Topological sort
symmetric
All arrows have a back arrow.

asymmetric (not symmetric)
There exists an arrow without a back arrow.

antisymmetric ("very" not symmetric)
No arrows have back arrows except self loops.
Formal definition:  
A relation $R$ on a set $X$ is antisymmetric if $\forall \ x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

Informal definition:  
If both an arrow and its reverse exist in an antisymmetric relation $R$, then that arrow must be a self loop (and, hence, it is its own reverse).

Alternate formal definition:  
A relation $R$ on a set $X$ is antisymmetric if $\forall \ (x, y) \in R$, either $x = y$ or $(y, x) \notin R$. 
A relation \( R \) on a set \( X \) is antisymmetric if \( \forall x, y \in X \), if \((x, y) \in R\) and \((y, x) \in R\), then \( x = y \).

**Ex 5.8.9.** Prove that \( | \) (divides) on \( \mathbb{N} \) is antisymmetric.

**Proof.** Suppose \( x, y \in \mathbb{N} \), \( x|y \), and \( y|x \) (that is, \((x, y), (y, x) \in |\)). By definition of divides, there exists \( i, j \in \mathbb{N} \) such that

\[
\begin{align*}
x &= i \cdot y \\
y &= j \cdot x
\end{align*}
\]

Then

\[
\begin{align*}
x &= i \cdot j \cdot x \quad \text{by substitution} \\
1 &= i \cdot j \quad \text{by cancellation} \\
i &= j = 1 \quad \text{by arithmetic} \\
x &= y \quad \text{by identity}
\end{align*}
\]

Therefore \( | \) is antisymmetric by definition. \( \square \)
Antisymmetry:
A relation \( R \) on a set \( X \) is *antisymmetric* if \( \forall x, y \in X, (x, y) \in R \) and \( (y, x) \in R \), then \( x = y \).

Partial order relation:
A *partial order relation* (or just *partial order*) is a relation that is reflexive, transitive, and antisymmetric.
A *strict partial order (relation)* is a relation that is irreflexive, transitive and antisymmetric.

Partially ordered set:
A *partially ordered set* or *poset* is a set together with a partial order on that set.
\[ R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\} \]
Comparable: $a \leq c, d \leq f, e \leq f, e \leq h, c \leq i$

Not comparable: $a$ and $b$; $d$ and $e$; $f$ and $h$

Maximal and greatest: $i$

Minimal: $a$ and $b$

No least

Everyday examples: Preparing a meal, writing a term paper, getting dressed
A partial order \( R \) on a set \( X \) is a total order if for all \( x, y \in X \), either \( x \preceq y \) or \( y \preceq x \), that is, \( x \) and \( y \) are comparable.

Standard example of a total order: \( \leq \).
A *partial order relation* (or just *partial order*) is a relation that is reflexive, transitive, and antisymmetric.

A partial order $R$ on a set $X$ is a *total order* if for all $x, y \in X$, either $x \leq y$ or $y \leq x$, that is, $x$ and $y$ are comparable.

A *topological sort* of a partial order $R$ is a total order that is a superset of $R$.

\[
| \, (\text{divides}) \quad \leq
\]

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$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}$

A topological sort for $R$: $R \cup \{(b, c)\}$, written as $a, b, c, d$

Another topological sort for $R$: $R \cup \{(c, b)\}$, written as $a, c, b, d$
For next time:

Pg 226: 5.8.(1-5)
Pg 231 5.9.(1 & 8)

Read 6.(1–3)