Chapter 7 outline:

- Introduction, function equality, and anonymous functions (last week Friday)
- Image and inverse images (Today)
- Function properties, composition, and applications to programming (Wednesday)
- Cardinality (Friday)
- Countability (Monday of Thanksgiving week, Nov 20)
- Review (Monday after Thanksgiving, Nov 27)
- Test 3, on Ch 6 & 7 (Wednesday, Nov 29)

Today:

- Review definitions from last time
- New definitions: image and inverse image
- Proofs
- Programming
A relation \( f \) from \( X \) to \( Y \) is a function (written \( f : X \to Y \)) if \( \forall x \in X \),
\[(1) \exists y \in Y \mid (x, y) \in f, \text{ and } (2) \forall y_1, y_2 \in Y, (x, y_1), (x, y_2) \in f \to y_1 = y_2 .\]

Not a function.  
(There’s a domain element that is related to two things.)

Not a function.  
(There’s a domain element that is not related to anything.)

A function.  
(It’s OK that two domain elements are related to the same thing and one codomain element has nothing related to it.)
Image

\[ F(A) = \{ y \in Y \mid \exists x \in A \text{ such that } f(x) = y \} \]

Inverse image

\[ F^{-1}(B) = \{ x \in X \mid f(x) \in B \} \]
Lemma 7.2. If \( f : X \to Y \), then \( F(\emptyset) = \emptyset \).

Lemma 7.3. If \( f : X \to Y \), \( A \subseteq X \), and \( A \neq \emptyset \), then \( F(A) \neq \emptyset \).

Lemma 7.4. If \( f : X \to Y \), then \( F^{-1}(\emptyset) = \emptyset \).

We might expect the following, but it’s not true:

Lemma XXXX. If \( f : X \to Y \), \( A \subseteq Y \), and \( A \neq \emptyset \), then \( F^{-1}(A) \neq \emptyset \).
Ex 7.4.1. If $A, B \subseteq X$, then $F(A \cap B) \subseteq F(A) \cap F(B)$. 
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

Consider this picture of $X$ and $Y$:
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $A, B \subseteq X$ and $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $A, B \subseteq X$ and $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \notin B$. By definition of image, $f(x) \in F(A)$.
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $A, B \subseteq X$ and $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \not\in B$. By definition of image, $f(x) \in F(A)$. So, also by definition of image, $f(x) \not\in F(B)$. Right?
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $A, B \subseteq X$ and $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \not\in B$. By definition of image, $f(x) \in F(A)$. So, also by definition of image, $f(x) \notin F(B)$. Right?

**NO!**
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

Let $X = \{x_1, x_2\}$, $Y = \{y\}$, $A = \{x_1\}$, and $B = \{x_2\}$.

Let $f = \{(x_1, y), (x_2, y)\}$.

Then $F(A - B) = F(\{x_1\} - \{x_2\}) = F(\{x_1\}) = \{y\}$.

Moreover, $F(A) - F(B) = \{y\} - \{y\} = \emptyset$.

So $F(A - B) \not\subseteq F(A) - F(B)$.
Ex 7.4.4. If $A \subseteq B \subseteq X$, then $F(B) = F(B - A) \cup F(A)$. 
Ex 7.4.6. If $A \subseteq B \subseteq Y$, then $F^{-1}(A) \subseteq F^{-1}(B)$.
Ex 7.4.7. If \( A, B \subseteq Y \), then \( F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B) \).
For next time:

Pg 342: 7.4.(2, 5, 8, 9, 10)

(Programming problems are with the next assignment)

Read 7.(6-8)

Take quiz