Chapter 7 outline:

- Introduction, function equality, and anonymous functions (last Friday)
- Image and inverse images (Today)
- Function properties, composition, and applications to programming (Wednesday)
- Cardinality (Friday)
- Countability (next week Monday)
- Review (Monday, Nov 29)
- Test 3, on Ch 6 & 7 (Wednesday, Dec 1)

Today:

- Review definitions from last time
- New definitions: image and inverse image
- Proofs
- Programming
A relation \( f \) from \( X \) to \( Y \) is a function (written \( f : X \rightarrow Y \)) if \( \forall x \in X \),

(1) \( \exists y \in Y \mid (x, y) \in f \), and (2) \( \forall y_1, y_2 \in Y \), \((x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2 \).

\[ (\text{Not a function.}) \]
\[ (\text{There's a domain element that is related to two things.}) \]

\[ (\text{Not a function.}) \]
\[ (\text{There's a domain element that is not related to anything.}) \]

\[ (\text{A function.}) \]
\[ (\text{It's OK that two domain elements are related to the same thing and one codomain element has nothing related to it.}) \]
**Image**

\[ F(A) = \{ y \in Y \mid \exists x \in A \text{ such that } f(x) = y \} \]

**Inverse image**

\[ F^{-1}(B) = \{ x \in X \mid f(x) \in B \} \]
Lemma 7.2. If $f : X \to Y$, then $F(\emptyset) = \emptyset$.

Lemma 7.3. If $f : X \to Y$, $A \subseteq X$, and $A \neq \emptyset$, then $F(A) \neq \emptyset$.

Lemma 7.4. If $f : X \to Y$, then $F^{-1}(\emptyset) = \emptyset$.

We might expect the following, but it's not true:

Lemma XXXXX. If $f : X \to Y$, $A \subseteq Y$, and $A \neq \emptyset$, then $F^{-1}(A) \neq \emptyset$. 
Ex 7.4.1. If $A, B \subseteq X$, then $F(A \cap B) \subseteq F(A) \cap F(B)$. 

[Diagram showing sets $A$, $B$, $F(A)$, $F(B)$, and their relationships.]
**Ex 7.4.3.** If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

Consider this picture of $X$ and $Y$:
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$. 
**Ex 7.4.3.** If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \notin B$. By definition of image, $f(x) \in F(A)$.
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

**Attempted proof.** Suppose $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \notin B$. By definition of image, $f(x) \in F(A)$.

So, also by definition of image, $f(x) \notin B$. Right?
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

Attempted proof. Suppose $y \in F(A - B)$. By definition of image, there exists $x \in A - B$ such that $f(x) = y$.

By definition of difference, $x \in A$, and $x \notin B$. By definition of image, $f(x) \in F(A)$. So, also by definition of image, $f(x) \notin B$. Right?

NO!
Ex 7.4.3. If $A, B \subseteq X$, then $F(A - B) \subseteq F(A) - F(B)$?

Let $X = \{x_1, x_2\}$, $Y = \{y\}$, $A = \{x_1\}$, and $B = \{x_2\}$.

Let $f = \{(x_1, y), (x_2, y)\}$.

Then $F(A - B) = F(\{x_1\} - \{x_2\}) = F(\{x_1\}) = \{y\}$.

Moreover, $F(A) - F(B) = \{y\} - \{y\} = \emptyset$.

So $F(A - B) \not\subseteq F(A) - F(B)$.
Ex 7.4.4. If $A \subseteq B \subseteq X$, then $F(B) = F(B - A) \cup F(A)$. 

[Diagram showing sets $X$, $B$, and $A$.]
Ex 7.4.6. If $A \subseteq B \subseteq Y$, then $F^{-1}(A) \subseteq F^{-1}(B)$.
Ex 7.4.7. If $A, B \subseteq Y$, then $F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$. 
For next time:
Pg 342: 7.4.(2, 5, 8, 9, 10)
(Programming problems are with the next assignment)
Read 7.(6-8)
Take quiz