Chapter 5 roadmap:

- Introduction to relations (Monday before break)
- Properties of relations (Wednesday and Friday before break)
- Transitive closure (Today)
- Partial order relations (Wednesday)
- Review for Test 2 (Friday)
- Test 2 on Chapters 4 & 5 (next week Monday)

Today:

- Review of relation properties
- An arithmetic on relations
- Computing whether a function is transitive
- Transitive closure
<table>
<thead>
<tr>
<th>Relation Type</th>
<th>Symbol</th>
<th>Description</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>A relation from one set to another</td>
<td>$R$</td>
<td>set of pairs subset of $X \times Y$</td>
<td>isEnrolledIn, isTaughtBy</td>
</tr>
<tr>
<td>A relation on a set</td>
<td>$R$</td>
<td>set of pairs subset of $X \times X$</td>
<td>eats, divides</td>
</tr>
<tr>
<td>The image of an element under a relation</td>
<td>$\mathcal{I}_R(a)$</td>
<td>set of things that $a$ is related to</td>
<td>classes Bob is enrolled in, numbers that 4 divides</td>
</tr>
<tr>
<td>The image of a set under a relation</td>
<td>$\mathcal{I}_R(A)$</td>
<td>set of things that things in $A$ are related to</td>
<td>classes Bob, Larry, or Alice are taking, numbers that 2, 3, or 5 divide</td>
</tr>
<tr>
<td>The inverse of a relation</td>
<td>$R^{-1}$</td>
<td>the arrows/pairs of $R$ reversed</td>
<td>hasOnRoster, teaches, isEatenBy, isDivisibleBy</td>
</tr>
<tr>
<td>The composition of two relations</td>
<td>$S \circ R$</td>
<td>two hops combined to one hop</td>
<td>hasAsProfessor, eatsSomethingThatEats</td>
</tr>
<tr>
<td>The identity relation on a set</td>
<td>$i_X$</td>
<td>everything is related only to itself</td>
<td>$\mathcal{I}_R(a) = {b \in Y \mid (a, b) \in R}$</td>
</tr>
<tr>
<td>$\mathcal{I}_R(a) = {b \in Y \mid (a, b) \in R}$</td>
<td></td>
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<tr>
<td>$\mathcal{I}_R(A) = {b \in Y \mid \exists a \in A \mid (a, b) \in R}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R^{-1} = {(b, a) \in Y \times X \mid (a, b) \in R}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S \circ R = {(a, c) \in X \times Z \mid \exists b \in Y \mid (a, b) \in R \land (b, c) \in S}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i_X = {(x, x) \mid x \in X}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Reflexivity

**Informal**
Everything is related to itself

**Formal**
\[ \forall x \in X, (x, x) \in R \]

**Visual**

**Examples**
\[ \subseteq, \leq, \geq, \equiv, i, \text{isAquaintedWith}, \text{waterVerticallyAligned} \]

### Symmetry

**All pairs are mutual**

**Formal**
\[ \forall x, y \in X, (x, y) \in R \rightarrow (y, x) \in R \]

**Visual**

**Examples**
\[ \equiv, \text{isOppositeOf}, \text{isOnSameRiver}, \text{isAquaintedWith} \]

### Transitivity

**Anything reachable by two hops is reachable by one hop**

**Formal**
\[ \forall x, y, z \in X, (x, y), (y, z) \in R \rightarrow (x, z) \in R \]

**Visual**

**Examples**
\[ <, \leq, >, \geq, \subseteq, \text{isTallerThan}, \text{isAncestorOf}, \text{isWestOf} \]
<table>
<thead>
<tr>
<th>Operators</th>
<th>$x + y$</th>
<th>$p \lor q$</th>
<th>$A \cup B$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-x$</td>
<td>$\sim p$</td>
<td>$\overline{A}$</td>
</tr>
<tr>
<td>Distribution</td>
<td>$x \cdot (y + z)$</td>
<td>$p \land (q \lor r)$</td>
<td>$A \cap (B \cup C)$</td>
</tr>
<tr>
<td></td>
<td>$= x \cdot y + x \cdot z$</td>
<td>$\equiv (p \land q) \lor (p \land r)$</td>
<td>$= (A \cap B) \cup (A \cap C)$</td>
</tr>
<tr>
<td>Identity</td>
<td>$x + 0 = x$</td>
<td>$p \lor T \equiv p$</td>
<td>$A \cup \emptyset = A$</td>
</tr>
<tr>
<td></td>
<td>$x \cdot 1 = x$</td>
<td>$p \land F \equiv p$</td>
<td>$A \cap U = A$</td>
</tr>
</tbody>
</table>
\[ S \circ R \]

\[ R^{-1} \]

\[ i_X \circ R = R \]

\[ R^2 = R \circ R \]
$R$  is one less than  eats  is parent of 

$R^2$  is two less than  eats something that eats  is grandparent of 

$R^3$  is three less than  eats something that eats something that eats  is great grandparent of 

???  $<$  gets nutrients from  is ancestor of
Definition of transitivity

Short form: \( \forall (x, y), (y, z) \in R, (x, z) \in R \)

Transform this to:

\[
\forall (x, y) \in R, \ \forall (w, z) \in R, \text{ if } y = w \text{ then } (x, z) \in R
\]
Definition of transitivity

Short form: $\forall (x, y), (y, z) \in R, (x, y) \in R$

Transform this to:

$\forall (x, y) \in R, \forall (w, z) \in R, \text{ if } y = w \text{ then } (x, z) \in R$
Definition of transitivity

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Definition of transitivity

Short form: \( \forall (x, y), (y, z) \in R, (x, y) \in R \)

Transform this to:

\[ \forall (x, y) \in R, \quad \forall (w, z) \in R, \quad \text{if } y = w \text{ then } (x, z) \in R \]
Definition of transitivity

Short form: $\forall (x, y), (y, z) \in R, (x, y) \in R$

Transform this to:

$\forall (x, y) \in R, \forall (w, z) \in R, \text{ if } y = w \text{ then } (x, z) \in R$
\{(1, 2), (2, 3), (5, 2), (1, 5), (2, 5), (1, 3)\}

\{\textcolor{red}{(1, 2)}, \textcolor{blue}{(2, 3)}, (5, 2), \textcolor{red}{(1, 5)}, \textcolor{blue}{(2, 5)}, (1, 3)\}

\{\textcolor{red}{(1, 2)}, \textcolor{blue}{(2, 3)}, (5, 2), (1, 5), \textcolor{blue}{(2, 5)}, (1, 3)\}

\{\textcolor{red}{(1, 2)}, \textcolor{blue}{(2, 3)}, (5, 2), (1, 5), (2, 5), (1, 3)\}
Computing transitivity is a $\forall\forall\exists$ problem

Our strategy is, for each pair $(x, y)$, walk through the whole (original) list. If the list

1. is empty, then true (vacuously)
2. begins with $(y, z)$ (that is, begins with $(w, z)$ where $y = w$), then search the whole (original) list for $(x, z)$.
   2.1 if found, keep searching
   2.2 if not found, then false
3. begins with $(w, z)$ for $w \neq y$, skip it and keep searching
<table>
<thead>
<tr>
<th>Domain</th>
<th>First relation</th>
<th>Second relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rivers</td>
<td>flows into</td>
<td>is tributary to</td>
</tr>
<tr>
<td></td>
<td>The Platte flows into the Missouri, and the Missouri flows into the Mississippi.</td>
<td>The Platte is a tributary to the Missouri; both the Platte and the Missouri are tributaries to the Mississippi.</td>
</tr>
<tr>
<td>People</td>
<td>is parent of</td>
<td>is ancestor of</td>
</tr>
<tr>
<td></td>
<td>Bill is Jane’s parent; Jane is Leroy’s parent</td>
<td>Bill is Jane’s ancestor; Leroy has both Jane and Bill as ancestors.</td>
</tr>
<tr>
<td>Domain</td>
<td>First relation</td>
<td>Second relation</td>
</tr>
<tr>
<td>--------</td>
<td>----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Animals</td>
<td><em>eats</em></td>
<td><em>derives nutrients from</em></td>
</tr>
<tr>
<td></td>
<td>Rabbit eats clover; coyote eats rabbit.</td>
<td>Coyote derives nutrients from rabbit; rabbit derives nutrients from clover; both coyote and rabbit ultimately derive nutrients from clover.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mathbb{Z}$</th>
<th><em>is one less than</em></th>
<th>$&lt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 is one less than 3; 3 is one less than 4</td>
<td>$2 &lt; 3; \ 3 &lt; 4; \ 2 &lt; 4.$</td>
<td></td>
</tr>
</tbody>
</table>
N Platte → S Platte → Platte → Canadian → Arkansas → Missouri → Mississippi → Allegheny → Monogahela → Tennessee → Ohio
If $R$ is a relation on $X$, then $R^T$ is the **transitive closure** of $R$ if

- $R^T$ is transitive
- $R \subseteq R^T$
- If $S$ is a transitive relation such that $R \subseteq S$, then $R^T \subseteq S$
Theorem 5.12 The transitive closure of a relation $R$ is unique.

Proof. Suppose $S$ and $T$ are relations fulfilling the requirements for being transitive closures of $R$. By items 1 and 2, $S$ is transitive and $R \subseteq S$, so by item 3, $T \subseteq S$. By items 1 and 2, $T$ is transitive and $R \subseteq T$, so by item 3, $S \subseteq T$. Therefore $S = T$ by the definition of set equality. □
Other closures:

**Ex 5.7.2** $R \cup i_A$ is the reflexive closure of $R$

**Ex 5.7.3.** $R \cup R^{-1}$ is the symmetric closure of $R$. (HW)
Ex 5.7.2 $R \cup i_A$ is the reflexive closure of $R$

**Proof.** Suppose $R$ is a relation on $A$.

[R $\cup i_A$ is reflexive:] Suppose $a \in A$. $(a, a) \in i_A$ by definition of identity relation. $(a, a) \in R \cup i_A$ by definition of union. Hence $R \cup i_A$ is reflexive by definition.

[R $\subseteq R \cup i_A$:] Suppose $(a, b) \in R$. Then $(a, b) \in R \cup i_A$ by definition of union. Hence $R \subseteq R \cup i_A$. (Alternately, we could have cited Exercise 4.2.1.)

[R $\cup i_A$ is the smallest such relation:] Suppose $S$ is a reflexive relation such that $R \subseteq S$. Suppose further $(a, b) \in R \cup i_A$. By definition of union, $(a, b) \in R$ or $(a, b) \in i_A$.

**Case 1:** Suppose $(a, b) \in R$. Then $(a, b) \in S$ by definition of subset (since we supposed $R \subseteq S$).

**Case 2:** Suppose $(a, b) \in i_A$. Then, by definition of identity relation, $a = b$. $(a, a) \in S$ by definition of reflexive (since we suppose $S$ is reflexive). $(a, b) \in S$ by substitution.

Either way, $(a, b) \in S$ and hence $R \cup i_A \subseteq S$ by definition of subset. Therefore, $R \cup i_A$ is the reflexive closure of $R$. □
Theorem 5.13 If $R$ is a relation on a set $A$, then

$$R^\infty = \bigcup_{i=1}^{\infty} R^i = \{(x, y) \mid \exists \ i \in \mathbb{N} \text{ such that } (x, y) \in R^i\}$$

is the transitive closure of $R$.

Proof. Suppose $R$ is a relation on a set $A$.
Suppose $a, b, c \in A$, $(a, b), (b, c) \in R^\infty$. By the definition of $R^\infty$, there exist $i, j \in \mathbb{N}$ such that $(a, b) \in R^i$ and $(b, c) \in R^j$. By the definition of relation composition and Exercise 5.7.4, $(a, c) \in R^j \circ R^i = R^{i+j}$. $R^{i+j} \subseteq R^\infty$ by the definition of $R^\infty$. By the definition of subset, $(a, c) \in R^\infty$. Hence, $R^\infty$ is transitive by definition.
Suppose $a, b \in A$ and $(a, b) \in R$. By the definition of $R^\infty$ (taking $i = 1$), $(a, b) \in R^\infty$, and so $R \subseteq R^\infty$, by definition of subset.
Suppose $S$ is a transitive relation on $A$ and $R \subseteq S$. Further suppose $(a, b) \in R^\infty$. Then, by definition of $R^\infty$, there exists $i \in \mathbb{N}$ such that $(a, b) \in R^i$. By Lemma 5.14, $(a, b) \in S$. Hence $R^\infty \subseteq S$ by definition of subset.
Therefore, $R^\infty$ is the transitive closure of $R$. □
For next time:

Pg 217: 5.6.(1 & 3)
Pg 222: 5.7.(3,4,5)

For Exercise 5.7.4, it should say \((S \circ R) \circ Q = S \circ (R \circ Q)\) instead of \((R \circ S) \circ Q = R \circ (S \circ Q)\).

Read 5.(8 & 9)