Continuous probability unit

- Introduction to continuous random variables (Wednesday)
- Continuous distributions 1: Uniform and Gaussian (**Today**)
- Continuous distributions 2: Exponential and Pareto (next week Monday)
- Jointly distributed random variables (next week Wednesday)

Today:

- Recap of continuous random variables, PDFs, etc
- The uniform distribution
- The Gaussian (normal) distribution

Let X be a random variable for probability function $P : \Omega \to \chi$. Then the **cumulative distribution function (CDF)** of X is $F : A \to [0, 1]$,

$$F(x) = P(X \leq x)$$

If χ is a countable subset of $\mathbb{R}$, then X is a **discrete** random variable, and the **probability mass function (PMF)** of X is

$$p(x) = P(X = x)$$

If χ is an uncountable subset of $\mathbb{R}$ (or just $\chi = \mathbb{R}$), then X is a **continuous** random variable, and the **probability density function (PDF)** of X is the derivative of F, that is,

$$f(x) = F'(x)$$

Moreover, for $a, b \in \chi$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Alternate definition. Let X be a random variable for probability function $P : \Omega \to \chi$. X is a **continuous** random variable if there exists a non-negative function $f : \chi \to \mathbb{R}$ such that for $a, b \in \chi$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Note that

$$P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

and

$$P(X = a) = \int_a^a f(x)\, dx = 0$$

and

$$f(x) = \lim_{\Delta x \to 0} \frac{P(x \leq X \leq x + \Delta x)}{\Delta x}$$

Analogous to the definition of expected value for discrete random variables, $E[X] = \sum_{x \in \chi} xp(x)$, we define the **expected value** of a continuous random variable X as

$$E[X] = \int_{-\infty}^{\infty} x\ f(x)\, dx$$

Theorem 1

For continuous random variable X and function $g : \mathbb{R} \to \mathbb{R}$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f(x)\, dx$$

Corollary 2

For continuous random variable X and $a, b \in \mathbb{R}$,

$$E[aX + b] = aE[X] + b$$

Let X be a discrete random variable with finite *chi*. Let $n = |\chi|$. Then X has a **uniform distribution** if

$$p(x) = \frac{1}{n}$$

Let X be a continuous random variable and let $\alpha, \beta \in \mathbb{R}$ (where $\alpha < \beta$) define an interval. Then X has a **uniform distribution** if its probability density function f is

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Buses arrive at a specified stop at 15-minute intervals starting at 7 AM. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that the passenger waits

a. less than 5 minutes;

b. more than 10 minutes.

Let X be a continuous random variable and let $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. Then X has a **normal** or **Gaussian** distribution if its probability density function f is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Moreover,

$$E[X] = \mu$$

and

$$Var(X) = \sigma^2$$

For next time:

Do the exercise at the bottom of the handout.