Discrete probability unit

- Random variables, introduction (last week Friday)
- Random variables, representing distributions
- Review (Wednesday)
- Test (Friday)
- Expected value and variance (Wednesday)
- Common discrete distributions (**Today**)
- Continuous random variables (next week)

Today:

- Finishing from last time. . .
  - Properties of expected value
  - Variance
- Discrete probability distributions
  - The uniform distribution
  - The Bernoulli distribution
  - The binomial distribution
  - The geometric distribution

For a random variable $X : \Omega \to \chi$ with probability mass function p, the **expected value** of X is

$$E[X] = \sum_{x \in \chi} x \; p(x)$$

Theorem 1

For any function f with domain χ,

$$E[f(X)] = \sum_{x \in \chi} f(x)\ p(x)$$

Theorem 2

For any $c \in \mathbb{R}$,

$$E[c] = \sum_{x \in \chi} c\ p(x) = c$$

Theorem 3

For any $a \in \mathbb{R}$,

$$E[aX] = \sum_{x \in \chi} ax\ p(x) = aE[X]$$

Theorem 4

For any $b \in \mathbb{R}$,

$$E[X + b] = \sum_{x \in \chi} (x + b)\ p(x) = E[X] + b$$

For random variable X with mean μ, the **variance** of X is

$$Var[X] = E\left[(X - \mu)^2\right]$$

Theorem 5

$Var(X) = E\left[X^2\right] - (E[X])^2$

Theorem 6

$Var(aX + b) = a^2 Var(X)$

Theorem 7

For random variable X *with mean* μ *and variance* σ^2,

$$E\left[(X - c)^2\right] = \sigma^2$$

Let $\chi = \{0, 1\}$, and let $p \in [0, 1]$. Then random variable X has a **Bernoulli distribution** if

$$
\begin{aligned}
p(x) &= \begin{cases} p & \text{if } x = 1 \\ (1-p) & \text{otherwise} \end{cases} \\
&= p^x (1-p)^{1-x}
\end{aligned}
$$

Let $\chi = \mathbb{Z}_{n+1} = \{0, 1, 2, \ldots n\}$ for some $n \in \mathbb{N}$ and $p \in [0, 1]$. Then random variable X has a **binomial distribution** if

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

A company produces widgets, each of which, independently, will be defective with probability .01. The widgets are sold in packages of 10, and the company offers a money-back guarantee that at most 1 widget is defective. What proportion of packages sold must the company replace?

Adapted from Ross, *A First Course in Probability*, 1997 , pg 145

Suppose a particular trait (such as eye color) seen in a species is classified on the basis of one pair of genes and suppose that d represents a dominant gene and r a recessive gene. Combinations dd and rd display the dominant trait, but rr does not. If two hybrid (rd) parents have a total of 4 children, what is the probability that 3 of the 4 children display the dominant trait?

Adapted from Ross, *A First Course in Probability*, 1997 , pg 147

Theorem 8

If X is a binomial random variable with parameters n and p and probability mass function $p()$, then

a. $E[X] = np$

b. $E[X^2] = np[(n-1)p + 1]$

c. $Var(X) = np(1-p)$

d. $\forall\ x \in [1, n], \frac{p(x)}{p(x-1)} = \frac{(n-x+1)p}{x(1-p)}$

Let $\chi = \mathbb{W}$ and $\lambda \in \mathbb{R}^+$. Then random variable X has a **Poisson distribution** if

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

Some examples of random variables that usually obey the Poisson probability law follow:

1. *The number of misprints on a page (or group of pages) of a book*
2. *The number of people in a community living to 100 years of age*
3. *The number of wrong telephone numbers that are dialed in a day*
4. *The number of packages of dog biscuits sold in a particular store each day*
5. *The number of customers entering a post office on a given day*
6. *The number of vacancies occurring during a year in the Supreme Court*
7. *The number of α-particles discharged in a fixed period of time from some radioactive material*

Ross, A First Course in Probability, 1997 , pg 154–155

Let $\chi = \mathbb{N}$ and $p \in [0,1]$. Then random variable X has a **geometric distribution** if

$$p(x) = (1-p)^{x-1}p$$

Theorem 9

If X is a geometric random variable with parameter p and probability mass function $p()$, then

a. $\sum_{x=1}^{\infty}(1-p)^{x-1}p = 1$

b. $E[X] = \frac{1}{p}$

c. $Var(X) = \frac{1-p}{p^2}$

An urn contains 10 white balls and 3 black balls. Balls are drawn one at a time, with replacement, until a black one is drawn.

1. What is the probability that exactly 4 draws are needed?
2. What is the probability that at least 6 draws are needed?

Adapted from Ross, *A First Course in Probability*, 1997 , pg 162.

For next time:

Do the exercise on the bottom of the handout.