Continuous probability unit

- Introduction to continuous random variables (last weekWednesday)
- Continuous distributions 1: Uniform (last week Friday)
- Continuous distributions 2: Gaussian and Exponential (Monday)
- Continuous distributions 3: Pareto (and Zipf) (Wednesday)
- Jointly distributed random variables (**Today**)

Today:

- Joint cumulative distribution functions and and probability mass/density functions
- Marginal probabilities
- Conditional probabilities and independence
- Covariance and correlation

Let X and Y be random variables. Then the **joint cumulative distribution function** of X and Y is

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

If X and Y are discrete, then their **joint probability mass function** is

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

Moreover,

$$F_{X,Y}(x,y) = \sum_{i \leq x} \sum_{j \leq y} p_{X,Y}(i,j)$$

Let X and Y be random variables. Then the **joint cumulative distribution function** of X and Y is

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

If X and Y are continuous, then their **joint probability density function** is $f_{X.Y}$ such that

$$F_{X,Y}(a,b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y) dx dy$$

Moreover,

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial a \, \partial b} F_{X,Y}(x,y)$$

and

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_{c}^{d} \int_{a}^{b} f(x,y) dx dy$$

Let Ω be a sample space and let A and B be events over that sample space. Then $P(A \cap B)$ is the **joint probability** of events A and B.

Suppose $A_0, A_1, \ldots A_{n-1}$ and $B_0, B_1, \ldots B_{m-1}$ are each partitions of sample space Ω. Then the joint probabilities $P(A_i \cap B_j)$ have the property that

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} P(A_i \cap B_j) = 1$$

In this context, the probability of an individual events, for example $P(A_i)$, is called a **marginal probability**, and can be computed by summing over the joint probabilities with the other partition:

$$P(A_i) = \sum_{j=0}^{m-1} P(A_i \cap B_j)$$

Let X and Y be jointly distributed random variables. The **marginal cumulative distribution function** of X is

$$F_X(a) = F_{X,Y}(a, \infty) = P(X \leq a)$$

If X and Y are discrete, the **marginal probability mass function** of X is

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

If X and Y are continuous, the **marginal probability density function** of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

Let A and B be events over a sample space. The **conditional probability** of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

(This is undefined if $P(B) = 0$.)

Moreover, A and B are **independent** if the following (equivalent) are true:

$$\begin{aligned} P(A \mid B) &= P(A) \\ P(B \mid A) &= P(B) \\ P(A \cap B) &= P(A)P(B) \end{aligned}$$

Let X and Y be jointly distributed random variables. Then, if X and Y are discrete, the **conditional probability mass function** of X, given that $Y = y$, is

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

If X and Y are continuous, the **conditional probability density function** of X, given that $Y = y$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

Let X and Y be jointly distributed random variables. Then X and Y are **independent** if for all $A, B \subseteq \mathbb{R}$,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

Or, equivalently, for all $a, b \in \mathbb{R}$,

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$$

Or, again equivalently, with the same a, b,

$$F_{X,Y}(a, b) = F_X(a)F_Y(b)$$

If X and Y are discrete, this is equivalent to, for all x, y

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

If X and Y are continuous, this is equivalent to, for all x, y

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Recall that the **variance** of random variable X is

$$Var(X) = E[(X - \mu)^2]$$

Let X and Y be jointly distributed random variables and let μ_X and μ_Y be their marginal means and σ_X and σ_Y be their marginal standard deviations (square roots of their variances) The **covariance** of X and Y is

$$Cov(X, Y) = E[(X - \mu_Y)(Y - \mu_Y)]$$

The **correlation** of X and Y is

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

If X and Y are independent, then $Cov(X, Y) = 0$, and hence so is $\rho(X, Y)$.

For next time:

Nothing. . .