Proof of Horner’s rule loop invariant \( y = \sum_{k=0}^{n-(i+1)} a_{k+i+1}x^k \):

**Init.** After 0 iterations, \( y = 0 \), \( i = n \) by assignment. So

\[
\sum_{k=0}^{n-(i+1)} a_{k+i+1} = \sum_{k=0}^{-1} a_{k+i+1}x^k = 0 = y
\]

**Maint.** Now, suppose this holds true after \( N \) iterations, that is

\[
y_{old} = \sum_{k=0}^{n-(i_{old}+1)} a_{k+i_{old}+1}x^k
\]

where \( y_{old} \) and \( i_{old} \) are \( y \) and \( i \) after \( N \) iterations. Likewise, let \( y_{new} \) and \( i_{new} \) be the values after \( N + 1 \) iterations.
By assignment $i_{\text{new}} = i_{\text{old}} - 1$. Then

$$y_{\text{new}} = a_{i_{\text{old}}} + x \cdot y_{\text{old}} \quad \text{by assignment}$$

$$= a_{i_{\text{old}}} + x \cdot \sum_{k=0}^{n-(i_{\text{old}}+1)} a_{k+i_{\text{old}}+1} x^k$$

$$= a_{i_{\text{new}}} + x \cdot \sum_{k=0}^{n-(i_{\text{new}}+2)} a_{k+i_{\text{new}}} x^k \quad \text{by substitution}$$

$$= a_{i_{\text{new}}} + \sum_{k=0}^{n-(i_{\text{new}}+1)} a_{k+i_{\text{new}}} x^{k+1} \quad \text{by distribution}$$

$$= a_{i_{\text{new}}} + \sum_{k=1}^{n-(i_{\text{new}}+1)} a_{k+i_{\text{new}}} x^k \quad \text{by change of variables}$$

$$= a_{0+i_{\text{new}}-1} x^0 + \sum_{k=1}^{n-(i_{\text{new}}+1)} a_{k+i_{\text{new}}} x^k$$

$$= \sum_{k=0}^{n-(i_{\text{new}}+1)} a_{k+i_{\text{new}}} x^k \quad \square$$
Formal definition of big-Theta:

$$\Theta(g(n)) = \{ f(n) \mid \exists c_1, c_2, n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}$$
Proof. Let $c_1 = \frac{1}{14}$, $c_2 = \frac{1}{2}$ and $n_0 = 7$. Suppose $n > 7$. Then

\[
\frac{1}{14} = \frac{1}{2} - \frac{3}{7} < \frac{1}{2}
\]

\[
\frac{1}{14} \leq \frac{1}{2} - \frac{3}{n} \leq \frac{1}{2}
\]

\[
\frac{n^2}{14} \leq \frac{1}{2} n^2 - 3n \leq \frac{n^2}{2}
\]

\[
c_1 n^2 \leq g(n) \leq c_2 n^2
\]

Therefore $g(n) = \Theta(n^2)$ by definition.
Theorem 3.1. For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Proof. Suppose $f = \Theta(g(n))$. Then, by definition of $\Theta$, there exist constants $c_1$, $c_2$, and $n_0$ such that for all $n \geq n_0$,

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

Let $c = c_2$. Then $0 \leq f(n) \leq c \cdot g(n)$, hence $f(n) = O(g(n))$ by definition. Similarly, let $c = c_1$. Then $0 \leq c \cdot g(n)$, hence $f(n) = \Omega(g(n))$.

Conversely, suppose $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. By the definitions, there exist $c$, and $n_1$ such that for all $n \geq n_1$, $0 \leq f(n) \leq c \cdot g(n)$, and there exist $c'$, and $n'_1$ such that for all $n \geq n'_1$, $0 \leq c' \cdot g(n) \leq f(n)$.

Let $c_1 = c'$, $c_2 = c$, and $n_0 = \max(n_1, n'_1)$. Hence $f(n) = \Theta(g(n))$. \qed
3.1-4. Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

To see that $2^{n+1} = O(2^n)$, note that $2^{n+1} = 2 \cdot 2^n$. Thus 2 is the constant we're looking for, and we're done.

Let's attempt a proof that $2^{2n} = O(2^n)$. Does $\exists c, n_0 | \forall n \le n_0, 2^{2n} \le c \cdot 2^n$? If so, then $2^n \cdot 2^n \le c \cdot 2^n \implies 2^n \le c$. . . which is impossible.
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$$2^n \cdot 2^n \leq c \cdot 2^n$$

$$2^n \leq c$$

...which is impossible.
3-1.d. If $k > d$, then $p(n) = o(n^k)$.

**Proof.** Suppose $k > d$ and suppose $c > 0$. Then

$$a_0 + a_1 n + \ldots + a_d n^d < a_x + a_x n + \ldots + a_x n^d \quad \text{where } a_x = \max(a_0, a_1, \ldots a_d)$$

$$< d \cdot a_x n^d$$

$$< c \cdot n^k$$

(see why I chose $a_x$ instead of $a_m$?)

if $n$ is big enough.

So, we want $d \cdot a_x < c \cdot n^{k-d}$. This holds as long as

$$n > \left( \frac{d \cdot a_x}{c} \right)^{\frac{1}{k-d}}$$