If \( f(n) = \omega(g(n)) \) then \( f(n) \neq O(g(n)) \).

**Proof.** Suppose \( f(n) = \omega(g(n)) \). Further suppose \( f(n) = O(g(n)) \).

By definition of big-Oh, there exists \( c \) and \( n_0 \) such that for all \( n \geq n_0 \), \( 0 \leq f(n) \leq cg(n) \). By definition of little-omega, there exists \( n_1 \) such that for all \( n \geq n_1 \), \( cg(n) < f(n) \).

Let \( n_2 = \max(n_0, n_1) \). Then \( cg(n_2) < f(n) \leq cg(n_2) \), which is a contradiction. Therefore \( f(n) \neq O(g(n)) \). \( \square \)
No algorithm to transform an arbitrary binary tree with $n$ comparable keys to a binary search tree with the same keys in expected time $o(n \lg n)$ exists.

**Proof.** Suppose such an algorithm for BST-building exists. Then, construct the following algorithm:

1. Given an array with of comparable keys, transform that array into a binary tree, such as by making it a long dangly list-like tree. This takes $\Theta(n)$ time.
2. Transform that binary tree into a BST using the supposed algorithm. This takes $o(n \lg n)$ time, according to our supposition.
3. Transform the BST into a sorted array by traversing the tree. This takes $\Theta(n)$ time.

This algorithm sorts the given array, and takes $\Theta(n) + o(n \lg n) + \Theta(n) = o(n \lg n)$ time. By Theorem 8.1, this is impossible. Therefore no such algorithm for BST-building exists. □
Consider the sequence of \texttt{delete()} operations from just after a defragmentation up through and including the next defragmenting. Let $m$ be size at the beginning of this sequence. There will be $\frac{m}{2} - 1$ operations that are constant time, and the, last, defragmenting operation costs $O(m)$. All together this costs $O(m)$. Using the aggregate method, we spread the $O(m)$ cost across the $\frac{m}{2}$ operations to consider them $O(1)$ each. Using the accounting method, charge each of the $\frac{m}{2} - 1$ non-defragmenting operations 3 units, one of the nulling of the position itself, and two for its contribution to the next defragmenting (one for a nulling, one for a filling).
A **field** is a set together with two binary operations satisfying certain properties, known as the **field axioms**. The real numbers with addition and multiplication is the canonical example, but everything in the FFT works for other fields (such as rational numbers and complex numbers).

A **polynomial** is a function of $x$ in the form

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

A polynomial with *degree* $k$ has $k + 1$ indices and least *degree bound* $k + 1$.

A polynomial with degree bound $n$ has degree no greater than $n − 1$ and has (at most) $n$ coefficients.

The product of two polynomials with (least common) degree bound $n$, has degree bound $2n − 1$. 
Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and $r$ such that
$A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

$$c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + cx^n = (q_0 + q_1x + \cdots + q_{n-1}x^{n-1})(x - x_0) + r$$

$$= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n - q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r$$

$$= (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots + (q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n$$

$$q_{n-1} = c_n$$
$$q_{n-2} = c_{n-1} + q_{n-1}x_0$$
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$$q_0 = c_1 + q_1x_0$$
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$$= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n - q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r$$

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Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and $r$ such that $A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

\[ c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + cx^n = (q_0 + q_1x + \cdots + q_{n-1}x^{n-1})(x - x_0) + r \]
\[ = q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n - q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r \]
\[ = (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots + (q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n \]

\[
\begin{align*}
q_{n-1} & = c_n \\
q_{n-2} & = c_{n-1} + q_{n-1}x_0 \\
& \vdots \\
q_0 & = c_1 + q_1x_0 \\
r & = c_0 + q_0x_0
\end{align*}
\]
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\]

\[
= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n
\]

\[
- q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r
\]

\[
= (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots
\]

\[
+ (q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n
\]

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q_{n-1} = c_n
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\vdots
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\]

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= q_0 x + q_1 x^2 + \cdots + q_{n-2} x^{n-1} + q_{n-1} x^n
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= (r - q_0 x_0) + (q_0 - q_1 x_0) x + \cdots
\]

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+ (q_{n-2} - q_{n-1} x_0) x^{n-1} + q_{n-1} x^n
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q_{n-2} = c_{n-1} + q_{n-1} x_0
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q_0 = c_1 + q_1 x_0
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r = c_0 + q_0 x_0
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Ex 30.1-2. To compute \( A(x_0) \), first find \( q(x) \) and \( r \) such that \( A(x) = q(x)(x - x_0) + r \). Then \( A(x_0) = r \).

\[
c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + cx^n = (q_0 + q_1x + \cdots + q_{n-1}x^{n-1})(x-x_0) + r
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= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n - q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r
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$$= (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots$$

$$+(q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n$$

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\[= q_0 x + q_1 x^2 + \cdots + q_{n-2} x^{n-1} + q_{n-1} x^n - q_0 x_0 - q_1 x_0 x - \cdots - q_{n-1} x_0 x^{n-1} + r\]
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\[
\begin{align*}
c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n & = \left(q_0 + q_1 x + \cdots + q_{n-1} x^{n-1}\right)(x - x_0) + r \\
& = q_0 x + q_1 x^2 + \cdots + q_{n-2} x^{n-1} + q_{n-1} x^n \\
& \quad - q_0 x_0 - q_1 x_0 - \cdots - q_{n-1} x_0 x^{n-1} + r \\
& = (r - q_0 x_0) + (q_0 - q_1 x_0) x + \cdots \\
& \quad + (q_{n-2} - q_{n-1} x_0) x^{n-1} + q_{n-1} x^n
\end{align*}
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\[ c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + cx^n = (q_0 + q_1x + \cdots + q_{n-1}x^{n-1})(x - x_0) + r \]

\[ = q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n - q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r \]

\[ = (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots + (q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n \]

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q_{n-1} &= c_n \\
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Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and $r$ such that $A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

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c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n = (q_0 + q_1 x + \cdots + q_{n-1} x^{n-1})(x - x_0) + r
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= q_0 x + q_1 x^2 + \cdots + q_{n-2} x^{n-1} + q_{n-1} x^n
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= (r - q_0 x_0) + (q_0 - q_1 x_0) x + \cdots
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q_{n-1} = c_n
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Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and $r$ such that $A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

$$c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n = (q_0 + q_1 x + \cdots + q_{n-1} x^{n-1})(x - x_0) + r$$

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\[
c_0 + c_1x + \cdots + c_{n-1}x^{n-1} + cx^n = (q_0 + q_1x + \cdots + q_{n-1}x^{n-1})(x - x_0) + r
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= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n
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\[
- q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r
\]

\[
= (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots
\]

\[
+ (q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n
\]

\[
q_{n-1} = c_n
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q_0 = c_1 + q_1x_0
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Ex 30.1-2. To compute $A(x_0)$, first find $q(x)$ and $r$ such that

$A(x) = q(x)(x - x_0) + r$. Then $A(x_0) = r$.

\[
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\]

\[
= q_0x + q_1x^2 + \cdots + q_{n-2}x^{n-1} + q_{n-1}x^n
\]

\[-q_0x_0 - q_1x_0x - \cdots - q_{n-1}x_0x^{n-1} + r
\]

\[
= (r - q_0x_0) + (q_0 - q_1x_0)x + \cdots
\]

\[+(q_{n-2} - q_{n-1}x_0)x^{n-1} + q_{n-1}x^n
\]

\[
q_{n-1} = c_n
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\]

\[\vdots\]

\[
q_0 = c_1 + q_1x_0
\]

\[
r = c_0 + q_0x_0
\]
Ex 30.1-4. Prove that \( n \) distinct point-value pairs are necessary to uniquely specify a polynomial of degree-bound \( n \). (Fewer than \( n \) distinct pairs fail to specify a unique polynomial.)

**Proof.** Let \( \{(x_0, y_0), \ldots (x_{n-2}, y_{n-2})\} \) be the set of points. WOLOG, assume no \( x_i = 0 \).

Add the point \((0, 0)\) to the set. By Theorem 30.1, this set specifies a unique \( n \)-degree-bound function \( A(x) \).

Alternately, add the point \((0, 1)\) to the set. By Theorem 30.1, this set specifies a unique \( n \)-degree-bound function \( B(x) \).

Since \( A(0) = 0 \neq 1 = B(0) \), it must be that \( A \neq B \). Therefore \( n \) points are necessary. \( \square \).
Given \( n \) points, \((x_0, y_0), \ldots (x_{n-1}, y_{n-1})\),

\[
A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}
\]